Quantum Mechanics on Multiply Connected Manifolds with Applications to Anyons in One and Two Dimensions

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1 Introduction

Physics defined on multiply connected manifolds is an old topic in theoretical physics. In the context of the path integral formalism it was studied by the first time by Schulman [1] in 1968 and rigourely formulated by Laidlaw and de Witt [2] and Dowker [3] in 1971.

The central point is that in a multiply connected manifold the paths have differents weights in the sum over histories and the problem -it does not exist in the "standard" quantum mechanics- is how to define a quantum theory taking in account this fact.

The problem of to how define a quantum theory on a topologically non-trivial manifold is not an academic problem but since it finds experimentally realizable systems such as the Aharanonov-Bohm effect and the anyons that could an explanation to the quantum Hall effect and, maybe, to high temperatures superconductivity.

The purpose of these lectures will be to explain some aspects of the quantum theory defined on multiply connected manifolds in the context of the path integral formulation and the applications that these ideas find in anyon physics in one and two dimensions.

Figure 1: The solenoid has infinite lenght and inside a constant magnetic field, it assumes also that the solenoid is impenetrable.

Let us start in section 2 explaning some examples that involve non-trivial topological aspects; this section does not involve calculations and the unique purpose is to introduce several useful concepts. In section 3, we introduce the formal definition of a multiply connected manifold. In Section 4, path integrals on arbitrary manifolds. In section 5, several applications are studied and in section 6, the conclusions are given.

2 Systems Defined on Non-Trivial Topological Manifolds

Let us start discussing the most popular example of a quantum theory defined on a multiply connected manifold, namely the Aharonov-Bohm effect (AB).

The AB effect consists in the following experimental arrangement

The important question is that theoretically one expect that in the screen interference lines can be observed such as in a diffraction experiment and the lines be dependent only on the magnetic field that is inside of the solenoid.

This example tell us that the electromagnetic potentials -that classically are unobservable- are quantum mechanically responsible for the observability of the interference pattern. The experimental question relative to the AB effect was only solved with a serie of experiments performed by Tonomura and collaborators in the beginning of the eighties[4] ... twenty five years after of the Aharonov and Bohm prediction.

From a theoretical point of view, one can see the AB effect as phenomenon that occurrs because the $\Re^2$ manifold (that is the plane where the paths live)

Figure 2: Some of the the infinites path of the electron

has a point removed (the point where the solenoid is) and, as a consequence, the configuration space of this system is $\Re^2 - \{0\}$.

The AB effect is an example of a mechanism that appears in many examples of recent physics, one of them is the problem of two-anyons.

In order to explain this problem, let us consider the motion of two non-relativistics particles moving on a plane. The motion is regular everywhere except in the point where the particles collide.

The colision condition in the point $x_1 = x_2$ is equivalent to the replacement

$$\Re^2 \rightarrow \Re^2 - \{0\}, \tag{1}$$

and, in consequence, the manifold (configuration space) has also a point removed.

One can see formally this example as a similar phenomenon to the AB effect, each particle has atached a flux-tube and in the case of the two particles, one can exactly map it to the AB effect.

Of course, there are questions that one should give an answer; what is the analogue of the magnetic field for the case of two particles?, how to implement technically this fact?, etc..

There is also another problem closely related with the previous ones, namely cosmic strings. The cosmic strings are solutions of the Einstein field equations when matter like point is present, the solution is

$$dS^2 = dt^2 - r^2 d\phi^{'2} - dr^2 - dz^2, \tag{2}$$

where ϕ' is the defect angle.

The cosmic strings are generally assumed to be singularities that remained after the formation of our universe and could be experimentally detectable.

The manifold when one project to a plane is again $\Re^2 - \{0\}$.

In the next sections we will try to formalize these examples developing appropriate techniques of calculation.

3 Rudiments of Homotopy Theory

The configuration space where one compute the propagator of a free particle is an example of a simply connected manifold. When we give two points of the manifold one can draw infinite paths between these points which are topologically equivalent.

The word 'deformable'has a technical connotation for a mathematical operation called homotopy transformation that we will define below[5].

The idea of a homotopy is the following; we will say that two curves are homotopically equivalent if it is possible to deform continuosly one into the other, in other words, two continuous applications f and g of the space X to the space Y, $g : X \rightarrow Y$ are homotopic (simbolically $f \sim g$) if it exists a continuous function $F : X \times I \rightarrow X$, where I is the closed interval $[0, 1]$, such that

$$F(x,t)|_{t=0} = f(x), \tag{3}$$

and

$$F(x,t)|_{t=1} = g(x), \tag{4}$$

with $(x, t) \in X$.

Is clear that the idea of homotopy define a class of equivalence between applications, *i.e.*,

i) $f \sim f$,

ii) $f \sim g \Rightarrow g \sim f$,

iii $f \sim g$ and $g \sim h \Rightarrow f \sim h$,

$\forall$ continuous function f, g and h.

If G is the space of the all continuous applications between X and Y, then a relation of equivalence has the property of decomposing the space in classes of equivalence or disjoint sets of functions homotopically equivalent.

If the functions g and f are homotopic, then they belong to the same class of homotopy, otherwise they are non-homotopically equivalent. We will

denote the homotopy class by $[\alpha]$ where the set $[\alpha]$ is the set of paths that are homotopically equivalent. In the case of the AB effect in fig. 2 the path 1 and 2 belong to the same class of homotopy.

Now, we will restrict only to the applications that are closed curves or loops; we will say that the loops α and β with basis in x_0 ($i.e$ the point where the extremes coincide) are equivalent if there exists a function $H : I \times I \rightarrow X$ such that $H(t,0) = \alpha$, $H(t^{'}) = \beta$ and $H(0,s) = H(1,s) = x_0$ $\forall s \in I$.

The function $H(s,t)$ is a homotopy, then if α, β and γ are loops with basis in $x_0 \in X$ then

$i)$ $\alpha \sim \alpha$, $i.e.$ any loop is equivalent itself.

$ii)$ If $\alpha \sim \beta \Rightarrow$ exist a homotopy $H : I \times I \rightarrow X$ with $H(t,0) = \alpha$, $H(t,1) = \beta$ and $H(0,s) = H(1,s) = x_0$.

$iii)$ $\alpha \sim \gamma$ if $\alpha \sim \beta$ and $\beta \sim \gamma$.

It is possible to define a homotopy $L(t,s)$ between α and γ as follows

$$L(t,s) = \left\{ \begin{array}{ll} H(s,2t) & 0 \leq t \leq \frac{1}{2} \\ H(s,2t-1) & \frac{1}{2} \leq t \leq 1 \end{array} \right. \tag{5}$$

and in consequence $\alpha \sim \gamma$.

One can think of a more tangible example considering the AB effect "joining" the two extremes in fig. 2. Observing fig. 2 one see that there are paths that can be classified by a "topological invariant" n (winding number).

In general the loops can be summed and the resulting sum is another loop that link $\sum n$ times the hole. The set of all loops is a group that is isomorphic to the integer group, however in order to implement this fact it is necessary to define the product of loops. The definition is the following; let α and β curves in X with $\alpha(1) = \beta(0)$, then the product $*$ of curves is

$$(\alpha * \beta)(t) = \left\{ \begin{array}{ll} H(2s,t) & 0 \leq s \leq \frac{1}{2} \\ H(2s-1,t) & \frac{1}{2} \leq s \leq 1 \end{array} \right. \tag{6}$$

Once these definitions are given, one can show that the set of all the homotopy classes of loops $\{[\alpha]\}$ with basis in x_0 of X is a group and is denoted by $\pi_1(X, x_0)$ and is formally equivalent to

$$\pi_1(X, x_0) = \{[\alpha]\}. \tag{7}$$

The set π_1 endowed of the operation $*$ defines the first homotopy group or fundamental group. The group π_1 is the first of an infinite set of $(n > 0)$

higher order homotopy groups, π_1 eventually can be non-abelian while the higher homotopy groups are all abelian.

4 Path Integrals on Multiply Connected Manifolds

In this section we will introduce the concept of path integrals on multiply connected manifolds. Let us start considering path integrals on simply connected manifolds and the most simple application, namely, the free non-relativistic particle; this example is a warm up exercise and will be useful when the path integral on a multiply connected manifold is considered at the end of this secton.

The idea of the path integral consist to sum on all the paths between the initial and final points A and B. The propagation amplitude between these points is equivalent to the computation of the formal sum

$$G[B,A] \sim \sum_{paths} \text{``}\textit{something}\text{''}. \tag{8}$$

The previous expression has two difficulties, firstly has the technical problem of how to define the sum between paths and second it has the physical problem of how to define "*something*". This second problem is equivalent to postulate the Schrödinger equation in the conventional quantum mechanics and it is equivalent to make the replacement

$$\text{``}\textit{something}\text{''} \to e^{\frac{i}{\hbar}S}, \tag{9}$$

where S is the action.

The first problem is technically more difficult and, in essence, their solution consists in replacing (for details see e.g [6])

$$\sum_{paths} e^{\frac{i}{\hbar}S} \to \int \prod_t dx(t) e^{\frac{i}{\hbar}\int dt L(x,\dot{x})}, \tag{10}$$

where $L(x,\dot{x})$ is the lagrangian of the system.

In general although one can give a discretization prescription, the physical quantities are well defined only giving correctly the boundary conditions.

Thus, if one is interested computing the propagator of a particle, we must give the boundary conditions

$$x(t_1) = x_1, \qquad\qquad x(t_2) = x_2, \tag{11}$$

and then the expression

$$G[x_2, x_1] = \int \mathcal{D}x(t) e^{\frac{i}{\hbar}S}, \tag{12}$$

plus (11) defines the propagation amplitude or propagator of the system, here $\mathcal{D}x(t) = \prod_t dx(t)$.

One can verify explicitly how work out these ideas considering explicitly the most simple example, namely the motion of a free non-relativistic particle in one dimension described by the lagrangian

$$L = \frac{1}{2}\dot{x}^2. \tag{13}$$

We are interested in computing the propagation amplitude $G[x_2, x_1]$ with the boundary conditions (11). In order to compute

$$G[x_2, x_1] = \int \mathcal{D}x(t) e^{\frac{i}{\hbar}\int_1^2 dt \frac{1}{2}\dot{x}^2}, \tag{14}$$

one start making the following change of variables

$$\begin{aligned} x(t) &= x_1 + \frac{\Delta x}{\Delta t}(t - t_1) + y(t) \\ &= x_{cl} + y(t), \end{aligned} \tag{15}$$

where x_{cl} is the solution of the classical solutions of equation of motion $\ddot{x} = 0$ and $y(t)$ is a quantum fluctuation that, by consistency, satisfy the boundary condition

$$y(t_1) = 0 = y(t_2). \tag{16}$$

When (15) is replaced in (14) one find that

$$G[x_2, x_1] = e^{i\frac{(\Delta x)^2}{2\Delta t}} \int \mathcal{D}y(t) e^{\frac{i}{2}\int_1^2 dt y(-\partial_t^2)y}. \tag{17}$$

The integral in y is gaussian and the result of the integration is

$$\det(-\partial_t^2)^{-\frac{1}{2}}, \tag{18}$$

Now one should compute the determinant, the procedure of calculation is the following: one start by solving the eingenvalue equation

$$\partial_t^2 \psi_n = \lambda_n \psi_n, \tag{19}$$

with Dirichlet boundary conditions and afterwards one use the formula

$$\det(-\partial_t^2) = \prod_n \lambda_n. \tag{20}$$

Using $\psi_n(t_1) = 0 = \psi_n(t_2)$, we find that $\lambda_n = (n\pi/\Delta t)^2$ and (20) becomes

$$\det(-\partial_t^2) = \prod_{-\infty}^{+\infty} (n\pi/\Delta t)^2, \tag{21}$$

however, (21) is divergent.

In order to give sense to the divergent expression (21), one regularize appropiately this product; firstly one observe that (21) has the general form $\prod an^b$, then one write

$$\prod an^b = e^{\sum_n logsn^b} = e^{\sum_n loga + b\sum_n logn}, \tag{22}$$

and using the Riemann ζ-function

$$\zeta(s) = \sum_n \frac{1}{n^s}, \tag{23}$$

one see that

$$\begin{aligned} \prod an^b &= e^{\log a \lim_{s\to 0} \cdot \sum_{n=1}^{n=\infty} n^{-s} + b \lim_{s\to 0} \frac{d}{ds} \sum n^{-s}} \\ &= e^{\log a\zeta(0) + b\zeta'(0)}. \end{aligned} \tag{24}$$

By analitic continuation one see that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2} log 2\pi$ and then

$$\prod an^b = a^{-\frac{1}{2}} (2\pi)^{\frac{b}{2}}, \tag{25}$$

so that (22) is simply $1/\Delta t$ and the propagator becomes

$$G[x_2, x_1] = \frac{1}{\sqrt{\Delta t}} e^{i(\Delta x)^2/\Delta t}, \tag{26}$$

that is the standard result.

In the previous problem we have assumed that the manifold is defined on

$$-\infty < x < \infty. \tag{27}$$

The next question is, what happens if the manifold has another topological structure such as a circle or a torus etc.?.

If the manifold has the topology of a circle, the boundary conditions (11) do not define completely the problem and one must modify (11) in the following way

$$\begin{aligned} x(t_1) &= x_1 \\ x(t_2) &= x_2 + 2n\pi, \end{aligned} \tag{28}$$

where n is an integer number (winding number).

A physical system described by the lagrangian

$$L = \frac{1}{2}\dot{x}^2, \tag{29}$$

with the boundary conditions (28), is called quantum rotator and it is the most simple example defined on a multiply connected manifold.

The strategy that we will follow below (and in essence is due to Schulman) is to solve this example in detail and afterwards to derive a general formula.

Let us start making the identification in (29) $x \to \phi$ and (28) becomes

$$\begin{aligned} \phi(t_1) &= \phi_1 \\ \phi(t_2) &= \phi_2 + 2n\pi. \end{aligned} \tag{30}$$

The propagation amplitude for this case becomes

$$G_n[\phi_2, \phi_1] = \int \mathcal{D}\phi(t) e^{\frac{i}{\hbar}\int_1^2 dt \frac{1}{2}\dot{\phi}^2}, \tag{31}$$

provided that the boundary condition (30) are assumed. When (31) is computed using (30), the propagation amplitude will depend on n, for this reason we have written $G_n[\phi_2, \phi_1]$.

One solve this problem in complete analogy with the free non-relativistic particle, in fact making the change of variables

$$\begin{aligned} \phi(t) &= \phi_1 + \frac{\Delta\phi + 2n\pi}{\Delta t}(t - t_1) + \psi(t) \\ &= \phi_{cl} + \psi(t), \end{aligned} \tag{32}$$

with ϕ_{cl} the classical solution of the equation of motion and, also by consistency, the quantum fluctuations satisfying $\psi(t_1) = 0 = \psi(t_2)$.

Replacing (32) in (31)

$$\begin{aligned} G_n[\phi_2, \phi_1] &= e^{\frac{i(\Delta\phi + 2n\pi)^2}{\Delta t}} \int \mathcal{D}\psi e^{i\int_{t_1}^{t_2} dt \frac{1}{2}\psi \partial_t{}^2 \psi} \\ &= e^{\frac{i(\Delta\phi + 2n\pi)^2}{\Delta t}} \det{(\partial_t^2)}^{-\frac{1}{2}}. \end{aligned} \tag{33}$$

The determinant is computed as in the free non-relativistic particle case and the result is Δt. Thus, (33) is

$$G_n[\phi_2, \phi_1] = \frac{1}{\sqrt{\Delta t}} e^{\frac{(\Delta\phi + 2n\pi)^2}{\Delta t}} \tag{34}$$

The expression (34) is the propagation amplitude for a fix homotopy class and, in consequence, the total propagation amplitude is

$$G[\phi_2, \phi_1] = \sum_{n=-\infty}^{n=+\infty} \Xi_n G_n[\phi_2, \phi_1], \tag{35}$$

where Ξ_n is a factor that we must determine. By invoking completness and unitarity of the Green function *i.e.*

$$\Xi_n^* \Xi_m = \Xi_{n+m}, \qquad \Xi_n^* \Xi_n = 1, \tag{36}$$

one find that Ξ_n must be $e^{n\delta}$ where δ is a phase.

Using the identity

$$\begin{aligned} \vartheta_3(\tau, z) &= \sum_{n=-\infty}^{n=+\infty} e^{2inz + i\pi n^2 \tau} \\ &= (-\tau)^{-\frac{1}{2}} e^{\frac{z^2}{i\pi\tau}} \vartheta_3(\frac{z}{\tau}, -\frac{1}{\tau}), \end{aligned} \tag{37}$$

then, one find the final expression

$$G[\phi_2, \phi_1] = \frac{1}{2\pi} \exp[i\frac{\delta\Delta\phi}{2\pi} - i\frac{\delta^2 \Delta}{8\pi^2}] \vartheta_3(\frac{\Delta\phi}{2} - \frac{\Delta t\delta}{4\pi}, -\frac{\Delta t}{2\pi}), \tag{38}$$

that is the result found by Schulman in 1968, althought the derivation given by him is slightly different.

Finally we will discuss briefly the formal derivation of the heuristic formula (16).

Firstly one should note the existence of two equivalent manifolds, $\mathcal{M}$ and their universal covering $\tilde{\mathcal{M}}$. $\mathcal{M}$ is a multiply connected manifold while $\tilde{\mathcal{M}}$ is simply connected.

Both manifolds are related by

$$\mathcal{M} = \frac{\tilde{\mathcal{M}}}{\Gamma}, \tag{39}$$

where Γ is a discrete group. By definition the quocient $\frac{\tilde{\mathcal{M}}}{\Gamma}$ is the set of all homotopy classes, *i.e*

$$\frac{\tilde{\mathcal{M}}}{\Gamma} = \{[x], x \in \tilde{\mathcal{M}}\}. \tag{40}$$

Then, two point x and $\tilde{x}$ are equivalent under Γ if there exists an element $g \in \Gamma$ such that

$$\tilde{x} = x.g. \tag{41}$$

In the case considered above, $\mathcal{M} = S^1$ and $\tilde{M} = \Re$ and the relation (41) is

$$\tilde{x} = x + 2\pi n, \tag{42}$$

while $\Gamma = \mathcal{Z}$ where $\mathcal{Z}$ is the integer group. Once this nomenclature is introduced one can define the path integral.

Let $\tilde{\psi}(\tilde{x})$ be the wave function on $\tilde{\mathcal{M}} = \Re$, this wave function is continuous and univalued then (41) becomes

$$\tilde{\psi}(\tilde{x}.g) = a(g).\tilde{\psi}(\tilde{x}), \forall g \in \mathcal{Z} \tag{43}$$

if one impose the normalization of the wave function

$$|a(g)| = 1, \tag{44}$$

then $a(g)$ is a phase that satisfies the following property; let the wave function with a well defined value, then there exists the pre-image $\tilde{x} = p^{-1}(x)$ of x with $\psi(x) = \tilde{\psi}(\tilde{x}_0)$; after a complete turn around in S^1, $\psi(x)$ takes another value and the pre-image will be different, say $\tilde{x}.g_1$ then

$$\tilde{\psi}(\tilde{x}_0) \rightarrow \tilde{\psi}(\tilde{x}_0.g_1) = a(g_1).\tilde{\psi}(\tilde{x}_0). \tag{45}$$

Giving another turn around one find

$$\begin{aligned}\tilde{\psi}(\tilde{x}_0.g_1) \to \tilde{\psi}(\tilde{x}_0.g_1 g_2) &= a(g_2)\tilde{\psi}(\tilde{x}_0.g_1) \\ &= a(g_1).a(g_2)\tilde{\psi}(\tilde{x}_0),\end{aligned} \tag{46}$$

and, as a consequence

$$\tilde{\psi}(\tilde{x}_0.g_1 g_2) = a(g_1).a(g_2)\tilde{\psi}(\tilde{x}_0), \tag{47}$$

thus

$$a(g_1.g_2) = a(g_1).a(g_2). \tag{48}$$

This last equation tell us, again, that $a(g)$ is a phase but also that there is a close relation between phase factors and the group. In the case at hand, the phase factor is an unitary irreducible representation of $\mathcal{Z}$.

The propagator in $\tilde{\mathcal{M}}$ is defined as usual,

$$\tilde{\psi}(\tilde{x}_1, \tilde{t}_1) = \int_{\tilde{\mathcal{M}}} d\tilde{x} \;\; \tilde{G}[\tilde{x}_2, \tilde{t}_2; \tilde{x}_1, \tilde{t}_1] \;\; \tilde{\psi}(\tilde{x}_2, \tilde{t}_2), \tag{49}$$

where $\tilde{G}$ is the propagator for univalued functions on $\mathcal{M}$ with space constituted by infinity copies of $\mathcal{M}$. Assuming continuity on the univalued functions, then one can write (49) as

$$\tilde{\psi}(\tilde{x}_1, \tilde{t}_1) = \sum_{\Re} \int_{\Re} d\tilde{x}_1 \;\; \tilde{G}[\tilde{x}_2, \tilde{t}_2; \tilde{x}_1, \tilde{t}_1]\psi(\tilde{x}_1.g), t^{'}, \tag{50}$$

denoting the arbitrary point $\tilde{x}^{'} \in \tilde{\mathcal{M}}$ by $\tilde{x}_0^{'} = \{\tilde{x}^{'} g\}$ where $x_0^{'}$ belong to a copy on $\mathcal{M}$ for some fundamental domain $\mathcal{M}_0$. Thus

$$\tilde{\psi}(\tilde{x}, \tilde{t}) = \sum_{g \in \mathcal{Z}} \int_{\tilde{\mathcal{M}}_0.g} d(\tilde{x}_0.g)\tilde{G}[\tilde{x}, \tilde{t}; \tilde{x}_0.g, \tilde{t}_1]\tilde{\psi}(\tilde{x}_0.g, t^{'}), \tag{51}$$

but as the copies are identical, the integration on any copy is the same on the fundamental domain, then we write

$$\tilde{\psi}(\tilde{x}, \tilde{t}) = \sum_{g \in \mathcal{Z}} \left\{ \int_{\tilde{\mathcal{M}}_0} d(\tilde{x})\tilde{G}[\tilde{x}, \tilde{t}; \tilde{x}_0.g, \tilde{t}_1] \right\} \tilde{\psi}(\tilde{x}_0.g, t^{'}), \tag{52}$$

or

$$\tilde{\psi}(\tilde{x}, \tilde{t}) = \int_{\tilde{\mathcal{M}}_0} d(\tilde{x}) \sum_{g \in \mathcal{Z}} \tilde{G}[\tilde{x}_2, \tilde{t}_2; \tilde{x}_0.g, \tilde{t}_1]\tilde{\psi}(\tilde{x}_0.g, t^{'}), \tag{53}$$

Following these arguments, then if $\psi(x,t)$ is the wave function in the point x, then exist a pre-image $\tilde{x}$ of x where $\tilde{\psi}(\tilde{x},t) = \psi(x,t)$. Furthermore, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are locally homeomorphic and, in consequence, $d\tilde{x} = dx$. Thus,

$$\psi(\tilde{x},\tilde{t}) = \int_{\mathcal{M}} dx G[\tilde{x},\tilde{t};x,t]\psi(x,t), \tag{54}$$

where

$$G[\tilde{x},\tilde{t};x,t] = \sum_{g\in\mathcal{Z}} \tilde{G}[\tilde{x}_0,\tilde{t}_0;\tilde{x},t^{'}]a(g), \tag{55}$$

with $x = p(\tilde{x}_0$ and $x = p(x_0)$. Making $x_0 \rightarrow x_0 g^{-1}$ and $\tilde{x}_0 \rightarrow \tilde{x}_0 g^{-1}$ and afterwards $g \rightarrow g^{-1}$, we arrive finally

$$G[\tilde{x},\tilde{t};x,t] = \sum_{g\in\mathcal{Z}} a(g^{-1})\tilde{G}[\tilde{x}_0,\tilde{t}_0;\tilde{x},t^{'}], \tag{56}$$

that is the standard formula for the propagator in a multiply connected manifold [2],[3]. Althought (56) was derived for a particular topology is a formula valid for general cases.

5 Applications

In this section we will apply the formulas derived in the above section to several problems such the AB effect including spin (and their relativistic extensions) and anyons.

The AB Effect In the AB effect the formal propagation amplitude is

$$G[x_2,x_1] = \sum_n \Xi_n \int_{(n)} \mathcal{D}x e^{iS_{free}}. \tag{57}$$

In order to compute (57) it is convenient to discretize as follow

$$\begin{aligned} G[x_2,x_1] &= \lim_{n\to\infty}\left(\frac{\rho}{i\pi\Delta t}\right)^n \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \prod_{k=1}^{n-1} dx_k dy_k \\ &\times \exp\left[i\rho\sum_{j=1}^{n}\left(\frac{(x_j-x_{j-1})^2}{\Delta t} + \frac{(y_j-y_{j-1})^2}{\Delta t}\right)\right], \end{aligned} \tag{58}$$

where $\rho = m/2, \Delta t = \frac{(t_2-t_1)}{m}$.

Using polar coordinates,

$$\begin{aligned} x_j &= r_j \cos\theta_j, \\ y_j &= r_j \sin\theta_j, \end{aligned} \tag{59}$$

and writing $dx_k dy_k = r_k dr_k d\theta_k$ and

$$(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 = r_j^2 + r_{j-1}^2 - r_j r_{j-1} \cos(\theta_j - \theta_{j-1}), \tag{60}$$

equation (58) becomes

$$\begin{aligned} G[x_2, x_1] &= \lim_{n\to\infty} \left(\frac{\rho}{i\pi\Delta t}\right)^n \int_0^\infty \dots \int_{-\pi}^{+\pi} \prod_{k=1}^{n-1} r_k dr_k d\theta_k \\ &\times \exp\left[\frac{i\rho}{\Delta t} \sum_{j=1}^{n} \left(r_j^2 + r_{j-1}^2 - r_j r_{j-1} \cos(\theta_j - \theta_{j-1})\right)\right]. \end{aligned} \tag{61}$$

Now we impose that the electron can turn around the solenoid, technically this is equivalent to impose the constraint

$$\phi + 2\pi m - \sum_{j=1}^{n} (\theta_j - \theta_{j-1}) \tag{62}$$

via a delta function, *i.e*

$$\begin{aligned} G[x_2, x_1] &= \lim_{n\to\infty} \left(\frac{\rho}{i\pi\Delta t}\right)^n \int_0^\infty \dots \int_{-\pi}^{+\pi} \prod_{k=1}^{n-1} r_k dr_k d\theta_k \delta[\phi + 2\pi m - \sum_{j=1}^{n} (\theta_j - \theta_{j-1})] \\ &\times \exp\left[\frac{i\rho}{\Delta t} \sum_{j=1}^{n} \left(r_j^2 + r_{j-1}^2 - r_j r_{j-1} \cos(\theta_j - \theta_{j-1})\right)\right], \end{aligned} \tag{63}$$

where ϕ is the angle between the source of electron, the center of the solenoid and the screen (see fig. 3).

Now one can exponenciate the δ function and after a tedious calculation we find

$$\begin{aligned} G &\quad [x_2, x_1]_m = \lim_{n\to\infty} \left(\frac{\rho}{i\pi\Delta t}\right)^n \int_0^\infty \dots \int_{-\pi}^{+\pi} \prod_{k=1}^{n-1} r_k dr_k \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{i\lambda(\phi + 2\pi m)} \\ \times &\quad \prod_{j=1}^{n} \exp\left[\frac{i\rho}{\Delta t}(r_j^2 + r_{j-1}^2)\right] \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \prod_{l=1}^{n-1} d\chi_l e^{-\frac{2i\rho}{\Delta t}(r_j r_{j-1} \cos\chi_j - i\lambda\chi_j)}, \end{aligned} \tag{64}$$

Figure 3: Here R and $R^{'}$ are the distances between the source and the screen to the centre of the solenoid.

In order to compute these integrals we use the asymptotic formula

$$\int_{-pi}^{+\pi} d\chi e^{i\lambda\chi + z\cos\chi} \rightarrow 2\pi I_{|\lambda|(z)} \tag{65}$$

valid in the limit $z \rightarrow \infty$.

Integrating in χ, and r

$$G[x_2, x_1]_m = \frac{\rho}{i\pi\Delta t} e^{i\frac{\rho}{\Delta t}(R^2 + R^{'2})} \int_{-\infty}^{\infty} d\lambda e^{i\lambda(\phi + 2\pi m)} I_{|\lambda|}(-2iRR^{'}/\Delta t), \tag{66}$$

thas is the the propagator for the m-th class of homotopy for the AB effect. This formula was obtained by first time by Inomata [7] and Gerry and Singh [8] in 1979 and simplified by Shiek[9] more recently.

The total propagator is

$$G[x_2, x_1] = \sum_{n=-\infty}^{n=\infty} e^{2\pi i\alpha m} G[x_2, x_1]_m, \tag{67}$$

where α is the magnetic flux. Replacing (66) in (67)

$$G[x_2, x_1] = \frac{\rho}{i\pi\Delta t} e^{i\frac{\rho}{\Delta t}(R^2 + R^{'2})} \sum_{-\infty}^{\infty} (-i)^{|m+\alpha|} e^{-i(m+\alpha)\phi} J_{|m+\alpha|}\left(\frac{2RR^{'}\rho}{\Delta t}\right). \tag{68}$$

Equation (68) has several "sub-applications", as was mentioned in section 2, the motion of two anyons is an example about it, in fact let us consider

the motion of 2 free particles in a plane, the lagrangian is

$$L = \frac{1}{2}\dot{\vec{x}}_1^2 + \frac{1}{2}\dot{\vec{x}}_2^2, \tag{69}$$

and defining relative coordinates and the center of mass as usual, one have

$$L = \frac{1}{2}\dot{\vec{x}}^2 + \frac{1}{2}\dot{\vec{X}}_{CM}^2 + \alpha\frac{d\Theta(\vec{x})}{dt}, \tag{70}$$

where $\vec{x} = \vec{x}_2 - \vec{x}_1$. The coordinate $\vec{X}_{CM}$ is the position of the center of mass, $\frac{d\Theta}{dt}$ is a topological that have been added by hand and that, classically, does not contributes to the equation of motion and Θ is the relative angle between the particles.

The partition function for this system becomes (the motion of the center of mass is trivially decoupled)

$$Z = \frac{1}{2}\int d^2\vec{r}\Big[< \vec{r}|e^{-\beta H_{rel}}|\vec{r} > + < \vec{r}|e^{-\beta H_{rel}}| - \vec{r} >\Big], \tag{71}$$

with

$$L_{rel} = \frac{1}{2}M\dot{\vec{r}}^2 + \alpha\dot{\theta}, \tag{72}$$

The brackets that appears in (71) are just the definition of the Green function and it was computed previously. However the propagator is divergent and one must regularize the expression

$$Z = \sum_{n=-\infty}^{n=\infty}\int_0^\infty dxe^{-x}I_{|n-\alpha|}(x). \tag{73}$$

In order to regularize one replace e^{-x} by $e^{\varepsilon x}$ and take the limit $\varepsilon \longrightarrow 0$ at the end of the calculation. The interested reader in the explicit calculations can see ref. [10], the final result is

$$\begin{aligned} F_\nu(a) &= \int_0^\infty dxe^{-ax}I_\nu(x) = \frac{1}{\sqrt{a^2-1}}\Big[a + \sqrt{a + \sqrt{a^2-1}}\Big]^{-\nu} \\ F_\nu(1+\varepsilon\quad) &\rightarrow \frac{1}{\sqrt{2\varepsilon}}\Big[1 + \sqrt{2}\varepsilon\Big]^{-\nu}. \end{aligned} \tag{74}$$

With these expressions in mind one can compute the second virial coefficient

$$B(\alpha, T) = 2\lambda_T^2 Z, \tag{75}$$

with $\lambda_T = (2\pi h^2/Mkt)^{\frac{1}{2}}$.

If one expand around the Fermi statistic $\alpha = 2j + 1 + \delta$ then one find

$$B(\alpha = 2j + 1 + \delta, T) = \frac{1}{4}\lambda_T^2 + 2\lambda_T^2. \tag{76}$$

More details about the calculation can be found *e.g* in the Lerda's book [11].

$\underline{Relativistic\ \ Aharanov - Bohm\ \ Effect}$

The relativistic extension of the AB effect is straightforward, but firstly one must define the path integral for a relativistic particle[12].

A relativistic particle is defined by the following lagrangian

$$L = \frac{1}{2N}\dot{x}^2 - \frac{1}{2}m^2 N, \tag{77}$$

where N is the einbein.

The classical symmetries of (77) are

$$\delta x^\mu = \epsilon \dot{x}^\mu, \qquad \delta N = (\epsilon \dot{N}), \tag{78}$$

The next step consists in to compute the propagation amplitude associated to (77), however it is not trivial because the relativistic particle is a generally covariant system and the propagator must be written á la Faddeev-Popov, *i.e.*

$$G[x_2, x_1] = \int \mathcal{D}N\mathcal{D}x^\mu \det(N)^{-1}\delta(f(N)) \det(\frac{\delta f(N)}{\delta\epsilon})\ e^{iS}. \tag{79}$$

This expression deserves some explanations. Firstly we have inserted the factor $\det(N)^{-1}$ by hand in order to have a functional measure invariant under general coordinate transformations; secondly the remaining factors are the usual terms of the Faddeev-Popov procedure, being $f(N) = 0$ the gauge condition.

An appropiate gauge condition for this problem is $\dot{N} = 0$ (proper-time gauge) and having in account the causality principle (79) becomes

$$G[x_2, x_1] = \int_0^\infty dT \int \mathcal{D}x^\mu\ \ e^{i\int_1^2 d\tau(\frac{1}{2N(0)}\dot{x}^2 - \frac{1}{2}m^2 N(0))}, \tag{80}$$

with $T = N(0)\Delta\tau$ and the boundary condition

$$x^{\mu}(\tau_1) = x_1^{\mu}, \qquad x^{\mu}(\tau_2) = x_2^{\mu}, \tag{81}$$

has been assumed. The formula (80) was found by Schwinger in 1951.

In order to compute (80) one repeat the arguments given in the non-relativistic case, making the change of variables

$$\begin{aligned} x^{\mu}(\tau) &= x_1^{\mu} + \frac{\Delta x^{\mu}}{\Delta\tau}(\tau - \tau_1) + y^{\mu}(\tau) \\ &= x_{cl}^{\mu} + y^{\mu}(\tau) \end{aligned} \tag{82}$$

where x_{cl}^{μ} is the classical solution of the equation of motion and y^{μ} is a quantum fluctuation that satisfies

$$y^{\mu}(\tau_1) = 0 = y^{\mu}(\tau_2). \tag{83}$$

Replacing (83) in (80) one find

$$\begin{aligned} G[x_2, x_1] &= \int_0^{\infty} dT T^{-D/2}\ e^{i\frac{(\Delta x)^2}{2T} - i\frac{m^2}{2}T}, \\ &= \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip.\Delta x}}{p^2 + m^2}, \end{aligned} \tag{84}$$

that is the expected result.

The next step is to apply these results in order to study the relativistic AB effect. The main idea is simple, one write the propagation amplitude for a relativistic particle as was discussed above and afterwards one separate the vector x^{μ} in components (x^0, x^1, x^2).

The main steps are the following:

i) Firstly instead of (80) one write

$$\begin{aligned} G[x_2, x_1] = \int_0^{\infty} & dT \quad e^{-i\frac{1}{2}m^2 N(0)} \times \\ & \times \quad \int \mathcal{D}x^0 e^{-i\int_1^2 d\tau \frac{1}{2N(0)}(\dot{x}^0)^2} \quad \int \mathcal{D}x e^{i\int_1^2 d\tau (\frac{1}{2N(0)}\dot{x}^2)}, \end{aligned} \tag{85}$$

then in (85) one can consider formally the integral in x_0 as an ordinary free non-relativistic particle with mass N_0^{-1} moving in a one dimensional space, the result of this integration is trivial

$$\frac{1}{\sqrt{T}}\ e^{-i\frac{\Delta x_0{}^2}{2T}}. \tag{86}$$

ii) The integral in the spatial coordinates is more complicated but one can map this problem to a non-relativistic problem with formal mass N_0^{-1}.

The final result is

$$\begin{aligned} G[x_2, x_1] &= \lim_{\epsilon \to 0} \sum_{n=-\infty}^{+\infty} \frac{(-i)^{|n+\alpha|}}{|n+\alpha} \, e^{-in+\alpha\phi} \int \frac{d^3p}{(2\pi)^3} e^{ip.\Delta x} \times \\ &\times \left(J_{|n+\alpha+1|}(\sqrt{RR'}\rho) + J_{|n+\alpha-1|}(\sqrt{RR'}\rho) \right) \times \\ &\times \left(K_{|n+\alpha+1|}(\sqrt{RR'}\rho) + K_{|n+\alpha-1|}(\sqrt{RR'}\rho) \right), \end{aligned} \tag{87}$$

with $\rho = \epsilon - (p^2 + m^2)$.

For details and other applications see [13].

6 Anyons in Two Dimensions

In this section we will discuss the idea of anyon from a more general point of view, but before let us consider a particular case known as Bose-Fermi Transmutation (BFt).

The idea due to Polyakov [14] consist in to take an spinning particle described by the action[15]

$$S = \int d\tau (m\sqrt{\dot{x}^2} - \frac{i}{2}\theta_\mu \dot{\theta}^\mu - \frac{i}{2}\theta_5 \dot{\theta}_5 + \lambda \theta^\mu \dot{x}_\mu + \sqrt{\dot{x}^2}\lambda\theta_5), \tag{88}$$

with θ_μ, θ_5 and λ fermionic variables. Then when one integrate the fermionic variables one find a bosonic description of a spinning particle or more precisely, an action like

$$S = \int d\tau (m\sqrt{\dot{x}^2} + topological \;\; invariant), \tag{89}$$

We will more precisely this result below.

In order to define appropriately the path integral one start defining the gauge condition

$$\theta_5 = 0, \tag{90}$$

that is consistent with the constraint $\theta^\mu \dot{x}_\mu = 0$.

The next step consists proposing the decomposition for the fermionic variable

$$\theta^\mu = n_1^\mu \kappa_1 + n_2^\mu \kappa_2 + e^\mu \kappa_e, \tag{91}$$

where n_1, n_2 and e are tri-vectors that satisfy

$$e^\mu = \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}, \qquad n_1^\mu . e_\mu = 0, \qquad n_i^\mu . n_{\mu j} = -\delta_{ij}. \tag{92}$$

The decomposition is equivalent to choose a Frenet-Serret frame where the n's are the normal vector and e is the vector tangent to the worldline.

The effective fermionic action is computed from

$$\begin{aligned} e^{iS(x)} &= \int \mathcal{D}\theta^\mu \mathcal{D}\theta_5 \mathcal{D}\lambda \ \delta(\theta_5) \ e^{iS} \\ &= \exp\left[im \int \tau \sqrt{\dot{x}^2} \right] \Phi \end{aligned} \tag{93}$$

where Φ is the Polyakov spin factor defined as

$$\begin{aligned} \Phi &= \int \mathcal{D}\kappa_1 \mathcal{D}\kappa_2 \exp\Big[\int d\tau \Big(-\frac{1}{2}(\kappa_1 \dot{\kappa}_1 + \kappa_2 \dot{\kappa}_2) + (n_1 . \dot{n}_2)\kappa_1 \kappa_2 \Big) \Big] \\ &= \det\Big[\frac{d}{d\tau} + (n_1 . \dot{n}_2) \Big]. \end{aligned} \tag{94}$$

The calculation of the determinant is straightforward[16],[17]

$$\det\Big[\frac{d}{d\tau} + (n_1 . \dot{n}_2) \Big] = e^{ic \int d\tau (n_1 . n_2)} \cos\Big[\frac{1}{2} \int d\tau (n_1 . n_2) \Big], \tag{95}$$

where c parametrize the differents possible regularizations. If one impose invariance under the interchanges n_1 and n_2 one find that $c = 0$ and the spin factor becomes

$$\Phi = \exp\Big[\frac{i}{2} \int dt (n_1 . \dot{n}_2) \Big] + \exp\Big[-\frac{i}{2} \int dt (n_1 . \dot{n}_2) \Big], \tag{96}$$

where the factors $\pm\frac{1}{2}$ denote the two possible spin states.

The next question is, how to generalize this result for other spin?. The answer can be obtained from Chern-Simons theories, let us start considering

a set of N relativistic particles minimally coupled to an abelian Chern-Simons field, the action is

$$S=\sum_{k=1}^{N}m\int d\tau\ \sqrt{\dot{x}_k^2}+\int d^3xJ_\mu A^\mu+\frac{1}{2\sigma}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{97}$$

where $J^\mu=\sum_{k=1}^{N}x^\mu\delta^{(3)}(x-x_k(\tau))$.

Then, one integrate the A_μ field and the result gives the effective action

$$S_{eff}=m\int d\tau\ \sqrt{\dot{x}^2}-\frac{\sigma}{2}\int d^3xd^3yJ^\mu(x)K_{\mu\nu}(x,y)J^\nu(y), \tag{98}$$

where $K_{\mu\nu}(x,y)$ is the inverse of the operator $\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$ and, of course, satisfy

$$\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho K_{\rho\sigma}(x,y)=\delta^\mu_\sigma\delta(x-y), \tag{99}$$

replacing J^μ in (98) one find that the non-local term becomes

$$-\frac{\sigma}{2}\sum_{i,j=1}^{N}I_{ij}, \tag{100}$$

where I_{ij} is

$$I_{ij}=\frac{1}{4\pi}\int dx_i^\mu dx_j^\nu\epsilon_{\mu\nu\rho}\frac{(x_i-x_j)^\rho}{|x_i-x_j|^3}. \tag{101}$$

For closed curves ij, I_{ij} becomes the linking number while for $i=j$ there are additional contributions in the one particle sector. These diagonal terms are computed[18] by a regularization as limit of non-diagonal terms, however the result can be dependent of the regularization. In order to perform this calculation one consider two curves infinitesimally near, I become

$$I=\frac{1}{4\pi}\int d\tau d\tau'\lim_{\epsilon\to 0}\epsilon_{\mu\nu\rho}\frac{dx_\varepsilon^\mu}{d\tau}\frac{(x_\varepsilon(\tau)-x(\tau'))^\nu}{|x_\varepsilon(\tau)-x(\tau')|^3}\frac{dx_\varepsilon^\rho}{d\tau'}, \tag{102}$$

with

$$x_\varepsilon(\tau)=x(\tau)+\varepsilon n(\tau), \tag{103}$$

the non-conmutativity between the limits $\varepsilon\to 0$ and the integration, imply

$$\begin{aligned} T &= \lim_{\varepsilon\to 0}L_\varepsilon - I \\ &= \frac{1}{2\pi}\int d\tau\ \epsilon_{\mu\nu\rho}e^\rho n^\nu\dot{n}^\rho, \end{aligned} \tag{104}$$

where $e^{\mu} = \frac{e^{\mu}}{|e|}$ is the normal principl vector. The quantity T is called the torsion of the curve and I is the self-linking of the curve.

The difference $T - L$ is denoted by $\mathcal{W}$ and is the writhing number number or cotorsion. Thus, the effect of the Chern-Simons field is to produce the interaction lagrangian

$$L_{int} = s\mathcal{W}, \tag{105}$$

where $s = \frac{\sigma}{4\pi}$ is the spin of the system. From this way one see that the BFt procedure is a particular case of a more general formulation coming from of a Chern-Simon contruction.

Other applications of anyons in two dimensions are discussed in the Fradkin's Lectures in this volume.

7 Anyons in One-Dimension

In this section we will discuss the possibility of anyons in one dimension. This possibility can be analized in complete analogy with the two dimensional case; in two dimensions there are anyons because have been points of the manifold. In one dimension one can repeat the same arguments as follow; let consider two non-relativistics particles moving on a line. For this system the configuration space is multiply connected (the real line minus the origin) because the point where the particles collide is singular. Classically the particles cannot go through each other, bouncing off elastically every time they meet. Thus, the action of this system is defined on the half-line [3,5], i.e.

$$S \;=\; \int_{t_1}^{t_2} dt\, \frac{1}{2}\dot{x}^2, \qquad\qquad (0 < x < \infty), \tag{106}$$

where x is the relative position of the two particles. As it is well known, the Hamiltonian associated to (106) is not self-adjoint on the naive Hilbert space because there is no conservation of probability at $x = 0$. The Hamiltonian for (106), however, can be made self-adjoint by adopting a class of boundary conditions for all the states in the Hilbert space of the form [19][20]

$$\psi^{'}(0) = \gamma\psi(0), \tag{107}$$

where γ is an arbitrary real parameter[1]

The calculation of the propagator to go from an initial relative position x_1 to a final one x_2 for the above problem gives [21], [22]

$$G_\gamma[x(t_2), x(t_1)] = G_0(x_2 - x_1) + G_0(x_2 + x_1) - 2\gamma \int_0^\infty d\lambda e^{-\gamma\lambda} G_0(x_2 + x_1 + \lambda), \tag{108}$$

where G_0 is the Green function for a free non-relativistic particle, *i.e*

$$G_0(x - y) = \frac{1}{\sqrt{2\pi i t}} e^{i(x-y)^2/2t}. \tag{109}$$

Although in one spatial dimension it is not possible to rotate particles, they can be exchanged and their "spin" and statistics can be determined by the (anti-) symmetry of the wave function. This (anti-) symmetry in turn depends on the values of the parameter γ.

This last fact can be seen by taking the limits $\gamma = 0$ and $\gamma = \infty$ of (3) [22]

$$G_{\gamma=0,\infty} = G_0(x_2 - x_1) \pm G_0(x_2 + x_1), \tag{110}$$

Under exchange of the positions of two particles in initial or final states, $G_{\gamma=0}$ is even and $G_{\gamma=\infty}$ is odd. Thus, for $\gamma = 0$ ($\gamma = \infty$) and the particles behave as bosons (fermions). The cases $0 < \gamma < \infty$ give particles with fractional spin and statistics [23].

The propagator (108) can also be obtained in the path integral representation, summing over all paths $-\infty < x(t) < \infty$, but in the presence of a repulsive potential $\gamma\delta(x)$. This problem was considered in[21] -[22] and the result is

$$G_\gamma[x(t_2), x(t_1)] \; = \; \int \mathcal{D}x(t) \; e^{iS}, \tag{111}$$

[1]There is an alternative approach to this problem. One could include in the classical configuration space the exchanged states where $x < 0$ is also permisible. The resulting system is described by the same action as (1) but with $x \neq 0$ instead of $x > 0$. In this configuration space the self-adjoint extension of the Hamiltonian imposes a condition that replaces (2), with two complex parameters $\gamma_\pm$ instead of one (γ). Here we shall not follows this approach. It is remarkable however that even if in our approach particle exchange is not included *ab initio*, quantum mechanics brings it in at the end.

with

$$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2}\dot{x}^2 + \gamma\delta(x(t))\right), \tag{112}$$

Here $\mathcal{D}x(t)$ is the usual functional measure. The potential term $\gamma\ \delta(x(t))$ can be interpreted as a semi-transprent barrier at $x = 0$ that allows the possibility of tunneling to the other side of the barrier. This is just another way of expressing the possibility of interchanging the (identical) particles.

It is also interesting to note here that although in (1+1) dimensions the rotation group is discrete and the definition of the spin is a matter of convention, one may nevertheless view the one-dimensional motion on the half-line as a radial motion with orbital angular momentum $l = 0$ [24] in a central potential. This gives rise to another possible definition of spin by taking the following representation for the δ-function

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\sqrt{\epsilon}}{x^2 + \epsilon}. \tag{113}$$

Making a series expansion around $\epsilon = 0$, the leading term $\gamma\sqrt{\epsilon}/x^2$ is analogous to the centrifugal potential for the radial equation in a spherically symmetric system, with $\sqrt{\epsilon}\gamma$ playing the role of an intrinsic angular momentum squared. Thus the spin of the system (s) can be defined by

$$s^2 = \sqrt{\epsilon}\gamma. \tag{114}$$

For real s (114) only makes sense when $\gamma > 0$. This definition is consistent with the bosonic limit $\gamma = 0$. For the fermionic case, the limit $\gamma = \infty$ mentioned above is to be interpreted as simultaneous with the limit $\epsilon \to 0$ so that $\sqrt{\epsilon}\gamma = 1/4$. It is in this sense that the non-relativistic quantum mechanics on the half-line describes one-dimensional anyons. However, the normalization $s = 1/2$ for fermions is conventional. One can extend these results for relativistics anyons although the calculations are more involved only we will give the final result for the propagator

$$\begin{aligned} G_\gamma[X(\tau_b), X(\tau_a)] &= G_0[X(\tau_b) - X(\tau_a)] + G_0[X(\tau_b) + X(\tau_a)] \\ &- 2\gamma \int_0^\infty d\lambda e^{-\gamma\lambda} G_0[X(\tau_b) + X(\tau_a) + \lambda]. \end{aligned} \tag{115}$$

The details are discussed in [25].

8 Conclusions

In these lectures we have discussed several aspects of quantum mechanics defined on non-trivial manifolds and, in particular, anyons in one and two dimensions. During the conference we discussed also other applications in one dimension and the connection between these results and bosonization[26].

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